Transformed proposal distributions

This note was originally written by Johan Lindström (Lund University, Sweden) and is now hosted at Umberto Picchini's research blog `https://umbertopicchini.wordpress.com/`.

A random walk update for a constrained variable $x \in I$ with density f, can be constructed by using an invertible transformation function, $g(x)$, such that $g(x) : \mathbb{R} \to I$. The resulting proposal is given by:

$$x^* = g\left(g^{-1}(x) + \epsilon\right), \qquad \epsilon \sim \mathcal{N}(0, \sigma^2).$$
$$q\left(x^* \mid x\right) = f_{\mathcal{N}(g^{-1}(x),\sigma^2)}\left(g^{-1}(x^*)\right) \cdot \frac{\partial g^{-1}(x^*)}{\partial x^*}.$$

Here the $f_{\mathcal{N}(g^{-1}(x),\sigma^2)}$ parts cancel in the acceptance probability and the adjustment due to change of variable becomes

$$\frac{q\left(x \mid x^*\right)}{q\left(x^* \mid x\right)} = \frac{\frac{\partial}{\partial x} g^{-1}(x)}{\frac{\partial}{\partial x^*} g^{-1}(x^*)}.$$

Some typical examples include:

Constrain	Functions			Acceptance
	$g(x)$	$g^{-1}(x)$	$\partial g^{-1}(x)$	$\frac{q(x\|x^*)}{q(x^*\|x)}$
$x > 0$	$\exp(x)$	$\log(x)$	x^{-1}	$\frac{1/x}{1/x^*} = \frac{x^*}{x}$
$x > a$	$\exp(x) + a$	$\log(x - a)$	$\frac{1}{x-a}$	$\frac{x^*-a}{x-a}$
$x < a$	$a - \exp(x)$	$\log(a - x)$	$\frac{-1}{a-x}$	$\frac{a-x^*}{a-x}$
$x \in [0, 1]$	$\frac{e^x}{e^x+1}$	$\log(x) - \log(1 - x)$	$\frac{1}{x(1-x)}$	$\frac{x^*(1-x^*)}{x(1-x)}$
$x \in [a, b]$	$\frac{be^x+a}{e^x+1}$	$\log(x - a) - \log(b - x)$	$\frac{b-a}{(b-x)(x-a)}$	$\frac{(x^*-a)(b-x^*)}{(x-a)(b-x)}$